Definition 9.10:

A vector function $\vec{F}(\vec{X},t)$ is said to satisfy a Lipschitz condition for $\vec{X}$ with a constant L, if

$$\left|\vec{F}(\vec{X}_1,t) - \vec{F}(\vec{X}_2,t)\right| \leq L\left|\vec{X}_1 - \vec{X}_2\right|,$$

where by $|\vec{X}|$ we denote the "norm" of the vector $\vec{X}$:

$$|\vec{X}| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$$

Theorem 9.2:

Let $\vec{F}(\vec{X},t)$ be continuous in t and satisfy a Lipschitz condition for $\vec{X}$ in the $(n+1)$-dim region R $|\vec{X}-\vec{X}_0| \leq b$, $|t-t_0| \leq a$.
Let M be the upper bound of $|\vec{F}|$ in R.
Then there exists a unique solution $\vec{X}(t)$ of the first order diff. eq.

$$\frac{d\vec{X}(t)}{dt} = \vec{F}(\vec{X},t), \quad \vec{X}(t_0) = \vec{X}_0,$$

defined over the interval $|t-t_0| \leq h = \min\left(a, \frac{b}{M}\right)$

Remark 9.5:

Consider the differential equation of nth order in one unknown

$$\frac{d^n x}{dt^n} = F\left(x^{(n-1)}, \ldots, x'(t), x; t\right) \qquad (1)$$

→ can be replaced by a system of differential equations of first order

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ &\vdots \\ \frac{dx_{n-1}}{dt} &= x_n \\ \frac{dx_n}{dt} &= F(x_1, x_2, \ldots, x_n; t) \end{aligned} \qquad (2)$$

It is easy to see that (1) and (2) are equivalent. Moreover, from Th. 9.2 we see that any unique solution is specified by

$$\vec{X}(t_0) = \vec{X}_0,$$

where $\vec{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$

Thus

$$\vec{X}_0 = \vec{X}(t_0) \overset{(2)}{=} \left(x_1(t_0), \left.\frac{dx_1}{dt}\right|_{t=t_0}, \dots, \left.\frac{dx_{n-1}}{dt}\right|_{t=t_0}\right)$$

$\longrightarrow$ We need the initial value of $x_1 = x$ and its first $(n-1)$ derivatives at initial time. One also says the solution-space is n-dimensional.

Let us now focus on the class of linear equations.

<u>Definition 9.11:</u>

The general form of a linear diff. eq. of nth order is

$$L(y) = a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y$$
$$= b(x).$$

The function L is called a "differential operator". The characteristic features of a "linear operator" L are

i) For any constants (C_1, C_2),

$$L(C_1 y_1 + C_2 y_2) = C_1 L(y_1) + C_2 L(y_2)$$

ii) For any given functions $p_1(x)$, $p_2(x)$, and the linear operators,

$$L_1(y) = a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \cdots + a_n(x)\,y$$

$$L_2(y) = b_0(x)\,y^{(n)} + b_1(x)\,y^{(n-1)} + \cdots + b_n(x)\,y,$$

the function

$$p_1 L_1 + p_2 L_2$$

defined by

$$(p_1L_1 + p_2L_2)(y) = p_1(x)L_1(y) + p_2(x)\,L_2(y)$$

$$= \left[p(x)\,a_0(x) + p_2(x)b_0(x)\right] y^{(n)} + \cdots$$

$$+ \left[p_1(x)\,a_n(x) + p_2(x)\,b_n(x)\right] y$$

is again a linear differential operator.

iii) Linear operators in general are subject to the "distributive law":

$$L(L_1 + L_2) = LL_1 + LL_2,$$

$$(L_1 + L_2)\,L = L_1L + L_2L.$$

iv) Linear operators with constant coefficients are commutative:

$$L_1 L_2 = L_2 L_1$$

Note: In general, linear operators with non-constant coefficients, are not commutative: namely,

$$L_1 L_2 \neq L_2 L_1.$$

For instance, let $L_1 = a(x)\frac{d}{dx}$, $L_2 = \frac{d}{dx}$.

$$\Rightarrow \quad L_1 L_2 = a(x)\frac{d^2}{dx^2} \neq L_2 L_1 = \frac{d}{dx}\left[a(x)\frac{d}{dx}\right].$$

Notation:

Henceforth, we will denote the differential operator $L(y) = y'$ by Dy, the operator $L(y) = y''$ by $D^2 y = D \circ D y$, where $\circ$ denotes composition of functions. More generally, we write

$$L(y) = y^{(n)} = D^n y.$$

$\rightarrow$ The general linear n-th order ODE then becomes

$$\left[a_0(x) D^n + a_1(x) D^{n-1} + \cdots + a_n(x) I\right](y) = b(x),$$

where I is the identity operator

$$I(y) = y = D^0(y).$$

Definition 9.12:

If $y_1, y_2, \dots, y_n$ are n functions which are $(n-1)$-times differentiable, then the "Wronskian" of these functions is the determinant

$$W = W(y_1, y_2, \dots, y_n)$$
$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Remark 9.6:

If $W(x_0) \neq 0$ for some point x_0, then $y_1, y_2, \dots, y_n$ are linearly indipendent. Indeed, consider the system of linear equations

$$c_1 y_1^{(i)}(x) + c_2 y_2^{(i)}(x) + \dots + c_n y_n^{(i)}(x) = 0, \quad (i = 0, \dots, n-1)$$

for all $x \in I$. If $W(x_0) \neq 0$ for some $x_0 \in I$, then we deduce $c_1 = c_2 = \dots = c_n = 0$.
Hence, the set of functions $y_1, \dots, y_n$ is linearly indipendent.

Proposition 9.5:

The Wronskian of n solutions of the n-th order linear ODE

$$L(y) = \left[D^n + a_1(x)D^{n-1} + \dots + a_n(x)\right](y) = 0,$$

is subject to the following first order ODE:

$$\frac{dW}{dx} = -a_1(x)W,$$

with solution

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)\,dt}.$$

$\longrightarrow$ the Wronkian is either identically zero, or vanishes nowhere.

Proof:

We give a proof here for the case of the second order equation only. Let

$$L(y_i) = y_i'' + a_1(x)y_i' + a_2 y_i = 0, \quad (i=1,2).$$

Note that

$$\frac{dW}{dx} = \frac{d}{dx}\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -(a_1y_1'+a_2y_1) & -(a_1y_2'+a_2y_2) \end{vmatrix}.$$

This yields

$$W'(x) = -a_1(x)W(x),$$

so that

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x_1} a_1(t)\,dt}.$$

$\square$

<u>Note:</u>

From the above we deduce the following:

i) if $W(x_0)=0$, then $W(x)=0 \ \forall \ x \in I$.

ii) if $W(x_0)>0$, then $W(x)>0 \ \forall \ x \in I$.

iii) if $W(x_0)<0$, then $W(x)<0 \ \forall \ x \in I$.

In other words, if the sign of the Wronskian $W(x)$ given by a set of functions $\{y_1, \dots, y_n\}$ changes on the interval I, then this set of functions cannot be a set of solutions for any linear homogeneous ODE.

<u>Proposition 9.6:</u>

If $y_1, y_2, \dots, y_n$ are solutions of the linear ODE $L(y)=0$, the following are equivalent:

i) $y_1, y_2, \ldots, y_n$ is a set of fundamental solutions, or a basis for the vector space $V = \operatorname{Ker}(L)$

ii) $y_1, y_2, \ldots, y_n$ are linearly independent.

iii) $W(y_1, y_2, \ldots, y_n) \neq 0$ at some point x_0

iv) $W(y_1, y_2, \ldots, y_n)$ is never zero.